Integration on Manifolds and Stokes' Theorem

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A Term Paper

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Abstract

Integration on manifolds is made possible using differential forms and orientability. In this term paper, we discuss differential k-forms and orientability, and how they naturally generalize integration on Euclidean spaces to integration on manifolds. Then we look at manifolds with boundary and discuss two classic theorems in differential geometry, Stokes' theorem and Green's theorem.

Introduction

In manifold theory, we use concepts from $\mathbb{R}^n$ to generalize these concepts to manifolds. Pullback is a recurring idea in manifolds where we make definitions for manifolds by taking $\mathbb{R}^n$ as reference. For example, to define differentiability on manifolds, we refer to differentiability on $\mathbb{R}^n$ which is already familiar to us. Using pullbacks make this possible. To make integration possible on manifolds, we introduce differential forms and we integrate forms on manifolds just like we integrate functions on $\mathbb{R}^n$.

In section 1, we introduce differential forms which generalize integration on $\mathbb{R}^n$ to integration on manifolds. In section 2, we discuss orientability of manifolds. Orientable manifolds are those to which we can give a consistent definition of "counterclockwise" and "clockwise". Not all manifolds are orientable. For example, a Möbius strip is non-orientable. In section 3, we discuss manifolds with boundary by taking the closed upper half space as a reference in the same way we take $\mathbb{R}^n$ as a reference for manifolds without boundary. An important result is that the boundary of an n-manifold is itself an $(n-1)$-manifold. In section 4, we define integration on manifolds. We cannot integrate on any manifold. We can only integrate differential n-forms on an oriented n-manifold under certain conditions. Once again, we use our understanding of integration on $\mathbb{R}^n$ to integrate on manifolds. In section 5, we introduce Stokes' theorem which relates the integration on a manifold with integration on its boundary. We deduce that when applied to a plane, Stokes' theorem reduces to Green's theorem, another classic theorem in differential geometry.

Knowledge of analysis (calculus) on $\mathbb{R}^n$ is assumed. Some knowledge of manifold theory such as differentiability on smooth manifolds and tangent spaces are assumed. Henceforth, we assume all manifolds to be smooth. Propositions and theorems are stated without proofs.

1 Differential forms

Informally, anything under an integral sign is a differential form. If f is a smooth real-valued function on a manifold M, its differential 1-form is the differential df defined by $(df)_p(X_p) = X_p f$ for any $p \in M$ and $X_p \in T_pM$. The cotangent space T_p^*M of M at p is the dual space of the tangent space T_pM, that is, $T_p^*M = \text{Hom}(T_pM, \mathbb{R})$. A covector at p is an element of the cotangent space T_p^*M. The cotangent bundle T^*M is the union of the cotangent spaces at all points of M.

A k-tensor on a vector space V is a k-linear function $f : V \times \cdots \times V \to \mathbb{R}$. The k-tensor f is alternating if for any permutation $\sigma \in S_k$,

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\text{sgn } \sigma) f(v_1, \ldots, v_k).$$

Alternating k-tensors on V form a vector space $A_k(V)$, also denoted by $\bigwedge^k(V^\vee)$. Alternating k-tensors on the tangent space T_pM form the vector space $\bigwedge^k(T_p^*M)$.

Definition 1.1 (differential k-form)**.** *A differential k-form or simply k-form is a function ω that assigns to each point $p \in M$ an alternating k-tensor $\omega_p \in \bigwedge^k(T_p^*M)$.*

A top form is an n-form where n is the dimension of the manifold. We are only concerned with top forms since they are the only forms that can be integrated.

Let $(U, x^1, \ldots, x^n)$ be a chart of M. A k-form on U is a linear combination $\omega = \sum a_I dx^I$, where $I \in \mathcal{J}_{k,n} = \{I = (i_1, \ldots, i_k \mid 1 \leq i_1 \leq \cdots < i_k \leq n)\}$ and the a_I are functions on U. Writing $\partial_i = \partial/\partial x^i$, we have the equality on U for $I, J \in \mathcal{J}_{k,n}$:

$$dx^I(\partial_{j_1}, \ldots, \partial_{j_k}) = \partial_J^I = \begin{cases} 1 & \text{for } I = J, \\ 0 & \text{for } I \neq J. \end{cases}$$

Proposition 1.2. *Let M be an n-manifold and $(U, x^1, \ldots, x^n)$ be a chart on M. Let $f, f^1, \ldots, f^n$ be smooth functions on U. Then*

$$df^1 \wedge \cdots \wedge df^n = \det\left[\frac{\partial f^j}{\partial f^i}\right] dx^1 \wedge \cdots \wedge dx^n.$$

Proposition 1.3. *A k-form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficient functions a_I are all smooth on U.*

Let $F : N \to M$ be a smooth map of manifolds. At each point $p \in N$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is a linear map and this induces a pullback map

$$F^* := (F_{*,p})^* : \bigwedge(T^*_{F(p)}M) \to \bigwedge(T^*_{F(p)}N).$$

If $\omega_{F(p)} \in \bigwedge(T^*_{F(p)}M)$, then its pullback $F^*(\omega_{F(p)})$ is the alternating k-tensor at p in N given by

$$F^*(\omega_{F(p)})(v_1, \ldots, v_k) = \omega_{F(p)}(F_{*,p}v_1, \ldots, F_{*,p}v_k), \quad v_i \in T_pN.$$

If ω is a k-form on M, then its pullback $F^*\omega$ is the k-form on N defined pointwise by $(F^*\omega)_p = F^*(\omega_{F(p)})$ for all $p \in N$.

2 Orientability

It is a common knowledge from single-variable calculus that reversing the limits of a definite integral reverses the sign of the integral. Along the real line, there are two orientations — going left and going right. In $\mathbb{R}^2$, there are two orientations again — going counterclockwise and going clockwise. If we consider only ordered bases, then we have a rule of defining orientations. Indeed, in $\mathbb{R}^3$, if $x = \{x^1, x^2, x^3\}$ and $y = \{y^1, y^2, y^3\}$ are two ordered bases, then there is a unique nonsingular 3×3 matrix A such that $x = Ay$ (take x and y as column vectors). Then we say x and y are of same orientation if $\det(A) > 0$. If $\det(A) < 0$, we say they are of opposite orientation. In fact, we can generalize this to higher dimensions and declare that $\mathbb{R}^n$ has only two orientations.

Let $x = \{x^1, \ldots, x^n\}$ and $y = \{y^1, \ldots, y^n\}$ be two ordered bases of $\mathbb{R}^n$. Then we say x and y are equivalent ordered bases and write $x \sim y$ if and only if $\det(A) = 0$. The matrix A is defined as above. It is easy to check that this defines an equivalence relation on the set of all ordered bases. Indeed, this partitions the set into two disjoint sets which define the two orientations of $\mathbb{R}^n$. This is generalized to any vector space. That is, every finite dimensional vector space has two orientations.

Consider an n-dimensional vector space V. Exploiting the fact that the space $\bigwedge^n(V^\vee)$ of n-covectors on V is one-dimensional, we can use an n-covector to specify orientation of V.

Lemma 2.1. *Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in a vector space V. Suppose*

$$u_j = \sum_{i=1}^{n} a^i_j v_i, \quad j = 1, \ldots, n,$$

for a matrix $A = [a^i_j]$ of real numbers. If β is an n-covector on V, then

$$\beta(u_1, \ldots, u_n) = (\det A)\beta(v_1, \ldots, v_n).$$

In the above lemma, suppose $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are ordered bases of V. See that $\beta(u_1, \ldots, u_n)$ and $\beta(v_1, \ldots, v_n)$ have the same sign if and only if $\det A > 0$ which is the same thing as saying that $\beta(u_1, \ldots, u_n)$ and $\beta(v_1, \ldots, v_n)$ are equivalent ordered bases. We say that β determines the orientation $(v_1, \ldots, v_n)$ if $\beta(v_1, \ldots, v_n) > 0$.

Let U be an open set of a smooth n-manifold M. Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ be two frames on U. If $Y_j = \sum_i a^i_j X_i$, then the two frames equivalent if and only if $\det(A) > 0$ at every point in U where $A = [a^i_j]$ is the change-of-basis matrix.

A pointwise orientation μ on M assigns to each $p \in M$ an orientation μ_p of the tangent space T_pM. The pointwise orientation μ is said to be continuous at $p \in M$ if there exists a neighborhood U of p on which μ is represented by a continuous frame. Of course, μ is continuous on M if it is continuous at every point $p \in M$.

Definition 2.2. *An orientation on M is a continuous pointwise orientation on M. An orientable manifold is one which has an orientation. A manifold together with an orientation is said to be oriented.*

It can be shown that, just as in the case of vector spaces, a connected orientable manifold M has exactly two orientations.

Lemma 2.3. *A pointwise orientation $[(X_1, \ldots, X_n)]$ on a manifold M is continuous if and only if each point $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which the function $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \cdots, X_n)$ is everywhere positive.*

Theorem 2.4. *A n-manifold M is orientable if and only if there exists a smooth nowhere-vanishing n-form on M.*

If ω and ω' are two nowhere-vanishing smooth n-forms on M, then $\omega = f\omega'$ for some nowhere-vanishing function f on M. If M is connected, such a function f is either everywhere positive or everywhere negative. This partitions the nowhere-vanishing smooth n-forms on an orientable manifold M into two equivalent classes by the equivalence relation

$$\omega \sim \omega' \text{ if and only } \omega = f\omega' \text{ with } f > 0.$$

Definition 2.5 (orientation form)**.** *An orientation form on M is a smooth nowhere-vanishing n-form ω such that $\omega(X_1, \dots, X_n) > 0$. We say that ω determines the orientation $[(X_1, \dots, X_n)]$.*

An oriented manifold is described by a pair $(M, [\omega])$, where $[\omega]$ is the equivalence class of an orientation form on M.

Definition 2.6 (oriented atlas)**.** *An atlas on M is oriented if for any two overlapping charts $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ of the atlas, the Jacobian $\det[\partial y^i/\partial x^j]$ is everywhere positive on $U \cap V$.*

It can be shown that a manifold M is orientable if and only if it has a oriented atlas.

3 Manifolds with boundary

We define the closed upper half space as follows:

$$\mathcal{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}.$$

We give $\mathcal{H}^n$ the subspace topology inherited from $\mathbb{R}^n$. We call the points with $x^n = 0$ the boundary points of $\mathcal{H}^n$ and the points with $x^n > 0$ the interior points of $\mathcal{H}^n$. Using a familiar notation from point-set topology, we may denote these sets by $\partial(\mathcal{H})^n$ and $(\mathcal{H}^n)^\circ$.

Definition 3.1. *Let $S \subset \mathbb{R}^n$ be a subset. A map $f : S \to \mathbb{R}^m$ is smooth at a point p in S if there exists a neighborhood U of p in $\mathbb{R}^n$ and a smooth map $\tilde{f} : U \to \mathbb{R}^m$ such that $\tilde{f} = f$ on $U \cap S$.*

Of course, we say f is smooth on S if it is smooth at each point of S.

Definition 3.2. *A topological space M is said to be locally $\mathcal{H}^n$ if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathcal{H}^n$.*

Definition 3.3. *A topological space M is said to be a topological n-manifold with boundary if it is second countable, Hausdorff topological space that is locally $\mathcal{H}^n$.*

Definition 3.4 (chart)**.** *Let M be a topological n-manifold with boundary and $n \geq 2$. Let $U \subset M$ be an open set. A chart on M is a pair (U, ϕ) where*

$$\phi : U \to \mathcal{H}^n$$

is a homeomorphism.

Clearly, $\phi(U)$ is open.

Definition 3.5 (smooth atlas)**.** *A smooth atlas is a collection $\{(U, \phi)\}$ of charts such that for any two charts (U, ϕ) and (V, ψ), the transition map*

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subset \mathcal{H}^n$$

is a diffeomorphism.

Definition 3.6 (interior point)**.** *An interior point p of M is such that for some chart (U, ϕ), the point $\phi(p)$ is an interior point of $\mathcal{H}^n$.*

We may similarly define

Definition 3.7 (boundary point)**.** *An exterior point p of M is such that for some chart (U, ϕ), the point $\phi(p)$ is a boundary point of $\mathcal{H}^n$.*

It is not hard to see that interior point and boundary point are well-defined, that is, they are independent of the charts we choose. Indeed, if (V, ψ) is another chart, then the diffeomorphism $\psi \circ \phi^{-1}$ maps $\phi(p)$ to $\psi(p)$, so that $\phi(p)$ and $\psi(p)$ are either both interior points or both boundary points.

Let M be a n-manifold with boundary ∂M. Let (U, ϕ) be a chart on M. Then we may restrict to $\phi' = \phi \mid_{U \cap \partial M}$ on the boundary. Since ϕ maps boundary to boundary,

$$\phi' : U \cap \partial M \to \partial \mathcal{H}^n = \mathbb{R}^{n-1}.$$

Let (U, ϕ) and (V, ψ) be two charts on M. Then

$$\psi' \circ (\phi')^{-1} : \phi'(U \cap V \cap \partial M) \to \psi'(U \cap V \cap \partial M)$$

is smooth. Therefore, an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M induces an atlas $\{(U_\alpha \cap \partial M, \phi_\alpha \mid_{U_\alpha \cap \partial U})\}$ for ∂M. This turns ∂M into a manifold of dimension $n - 1$ without boundary.

Let U and V be two neighborhoods of p in M. Two functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are said to be equivalent if they agree on some neighborhood of p contained in $U \cap V$. A germ of smooth functions at p is an equivalence class of such functions. It is customary to denote $[f]$, an equivalence class, by simply f. The set $C_p^\infty(M)$ of germs of smooth functions at p is an $\mathbb{R}$-algebra with the usual addition, multiplication, and scalar multiplication. The tangent space T_pM is defined to the vector space of all point-derivations on $C_p^\infty(M)$. The cotangent T_p^*M is the dual of the tangent space, that is, $T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R})$.

Differential k-forms on M defined to be sections of the vector bundle $\bigwedge^K(T^*M)$. A k-form is smooth if it is smooth as a section of $\bigwedge^K(T^*M)$. An orientation on an n-manifold M with boundary is a continuous pointwise orientation on M.

As in manifolds without boundary, the orientability of a manifold with boundary is equivalent to the existence of a smooth nowhere-vanishing top form and to the existence of an oriented atlas.

Definition 3.8. *Let $p \in \partial M$. A tangent vector $X_p \in T_pM$ is said to be inward-pointing if $X_p \notin T_p(\partial M)$ and there are a positive real number ϵ and a curve $c : [0, \epsilon] \to M$ such that $c(0) = p$, $c((0, \epsilon)) \subset M^\circ$, and $c'(0) = X_p$. A vector $X_p \in T_pM$ is outward-pointing if $-X_p$ is inward-pointing.*

A vector field along ∂M is a function X that assigns to each point p in ∂M a vector $X_p \in T_pM$. If $(U, x^1, \ldots, x^n)$ is a neighborhood of p in M, we may write

$$X_q = \sum_i a^i(q) \frac{\partial}{\partial x^i}\bigg|_q, \quad q \in \partial M.$$

Smoothness of X along ∂M at $p \in M$ is defined in terms of smoothness of the functions a^i on ∂M. A vector x_p is outward-pointing if and only if $a^n(p) < 0$.

Proposition 3.9. *On a manifold M with boundary ∂M, there is a smooth outward-pointing vector field along ∂M.*

It can be shown that the boundary ∂M of an oriented n-manifold M is orientable.

Proposition 3.10. *Let M be an oriented n-manifold with boundary. Let p be a point of the boundary ∂M and X_p be an outward-pointing vector in T_pM. An ordered basis $(v_1, \ldots, v_{n-1})$ for $T_p(\partial M)$ represents the boundary orientation at p if and only if the ordered basis $(X_p, v_1, \ldots, v_{n-1})$ for T_pM represents the orientation on M at p.*

4 Integration on manifolds

Integration on manifolds is quite different from integration on Euclidean spaces. For one, we can only integrate forms, not functions. For a n-manifold, we can only integrate n-forms which must have compact supprot. Also, M has to be oriented.

Let M be an oriented n-manifold whose orientation is given by an oriented atlas $\{U_\alpha, \phi_\alpha\}$. Let (U, ϕ) be a chart in this atlas. Let ω be an n-form with compact support on U. Since $\phi : U \to \phi(U)$ is a diffeomorphism, $(\phi^{-1})^*\omega$ is an n-form with compact support on the open subset $\phi(U) \subset \mathbb{R}^n$. We define the integral

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^*\omega.$$

It is easy to check that the above integral is well-defined, that is, it is independent of the choice of the chart. Linearity of the integral in Euclidean spaces implies that

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau.$$

If we choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$ of M, the sum

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is finite. It can be checked that the support of $\rho_\alpha\omega$ is compact so that the integral $\int_{U_\alpha} \rho_\alpha\omega$ is defined. We now define integral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega.$$

It can be checked that the integral defined above is well-defined, that is, it is independent of the choices of oriented atlas and partition of unity.

The following proposition shows that the sign of an integral over M is reversed upon reversing the orientation of M.

Proposition 4.1. *Let ω be an n-form with compact support on an oriented n-manifold M. Let $-M$ be the same manifold but with the opposite orientation. Then*

$$\int_{-M} \omega = -\int_M \omega.$$

The integration defined above for manifolds can be extended to oriented manifolds with boundary in a similar way.

5 Stokes' theorem and Green's theorem

Let M be an oriented n-manifold with boundary. The boundary ∂M is given the boundary orientation. Let $i : \partial M \hookrightarrow M$ be the inclusion map. Let ω be an $(n-1)$-form on M. We write $\int_{\partial M} \omega$ instead of $\int_{\partial M} i^*\omega$.

Theorem 5.1 (Stokes' theorem)**.** *If ω is any smooth $(n-1)$-form with compact support on the oriented n-manifold M,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Considering the special case of $\mathbb{R}^2$ gives us an interesting result. Let $D \subset \mathbb{R}^2$ be some plane region. Let M be D and ω be the 1-form $Pdx + Qdy$ on D. Then

$$\int_{\partial D} Pdx + Qdy = \int_{\partial D} \omega = \int_D \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy = \int_D \left(\frac{\partial D}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

This is the well-known Green's theorem. Therefore, in $\mathbb{R}^2$, Stokes' theorem reduces to Green's theorem. For completeness, we state the theorem as follows.

Theorem 5.2 (Green's theorem)**.** *Let $D \subset \mathbb{R}^2$ be some plane region. Let P and Q be smooth functions on D. Then*

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial D}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

The divergence theorem and the fundamental theorem of calculus may also be stated as special cases of Stokes' theorem.

References

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